Counting and Binomial Coefficients

Reference Counting Techniques

Objectives

- Cardinality of finite sets
- Factorials
- Reference counting formulas
- Permutations and arrangements
- Binomial coefficients
- Pascal's triangle
- Binomial theorem

Cardinality

Let E be a finite set.

$$\mathrm{Card}(E)$$

is the number of elements of E.

Example:

$$E = \{1, 2, 3, 4, 5\} \qquad \Rightarrow \qquad \mathrm{Card}(E) = 5.$$

Cardinality Properties

If A and B are finite sets, then

$$\mathrm{Card}(A \cup B) = \mathrm{Card}(A) + \mathrm{Card}(B) - \mathrm{Card}(A \cap B).$$

If $A \cap B = \varnothing$, then

$$\mathrm{Card}(A \cup B) = \mathrm{Card}(A) + \mathrm{Card}(B).$$

Factorial

For $n \geq 1$,

$$n! = n(n-1)\cdots 2 \cdot 1.$$

By convention,

$$0! = 1.$$

Example:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

Number of Functions

Let

$$\mathrm{Card}(E) = p, \qquad \mathrm{Card}(F) = n.$$

The number of functions from E to F is

$$\boxed{n^p}.$$

Each element of E has n possible images.

Example: Functions

How many functions are there from a set of 3 elements to a set of 4 elements?

$$n^p = 4^3 = 64.$$

$$\boxed{64}$$

Permutations

A permutation is an ordering of all elements.

The number of permutations of n elements is

$$\boxed{n!}.$$

Example:

$$4! = 24.$$

Arrangements Without Repetition

Choosing and ordering p elements from n elements:

$$\boxed{A_n^p = \frac{n!}{(n-p)!}}$$

Example:

$$A_{10}^3 = 10 \times 9 \times 8 = 720.$$

Remark:

$$A_n^p$$

also counts the number of injective functions from a set of cardinality p to a set of cardinality n.

Injective Functions

Let

$$\mathrm{Card}(E) = p, \qquad \mathrm{Card}(F) = n, \qquad p \leq n.$$

The number of injective functions

$$f : E \to F$$

is

$$\boxed{A_n^p = n(n-1)\cdots(n-p+1) = \frac{n!}{(n-p)!}.}$$

This is also the number of ordered selections of p distinct elements from a set of size n.

Example: Injective Functions

How many injective functions are there from

$$E = \{1, 2, 3\}$$

to

$$F = \{a, b, c, d, e\}?$$

Since

$$|E| = 3, \qquad |F| = 5,$$

we obtain

$$A_5^3 = 5 \times 4 \times 3 = 60.$$

60

Combinations

Choosing p elements from n elements without order:

$$\boxed{\binom{n}{p} = \frac{n!}{(n-p)!p!}}$$

Example:

$$\binom{10}{3} = 120.$$

Binomial Coefficients

For $0 \leq k \leq n$,

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!k!}}$$

It counts the number of subsets with k elements from a set with n elements.

Properties

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's Triangle

$n \backslash k$	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

Each number is the sum of the two numbers above it.

Binomial Theorem

For every $n \in \mathbb{N}$,

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

Example

Expand:

$$(x+y)^4.$$

Using Pascal's triangle:

$$1,\ 4,\ 6,\ 4,\ 1$$

we get

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

How to Choose the Formula

Ask yourself:

1. Does order matter?
2. Is repetition allowed?
3. Are we choosing all elements or only some?

Order matters $\Rightarrow A_n^p$ or $n!$

Order does not matter $\Rightarrow \binom{n}{p}$

Formula Table

Situation	Formula
Functions $E \to F$	n^p
Permutations of n objects	$n!$
Ordered choice of p objects	$A_n^p = \frac{n!}{(n-p)!}$
Unordered choice of p objects	$\binom{n}{p}$

Summary

- $\mathrm{Card}(E)$ counts elements.
- $n!$ counts permutations.
- n^p counts functions from a set of size p to a set of size n.
- A_n^p counts ordered choices without repetition.
- $\binom{n}{p}$ counts unordered choices.
- Pascal's formula:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$